Optimal robust bounds for variance options

Alexander Cox[1] Jiajie Wang[2]

[1]University of Bath

[2]Università di Roma 'La Sapienza'

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Financial Setting

- Option priced on an asset S_t
- Dynamics of S_t unspecified, but suppose paths are continuous, and we see prices of call options at all strikes K and at maturity time T
- Assume for simplicity that all prices are discounted — this won't affect our main results
- Under risk-neutral measure, S_t should be a (local-)martingale, and we can recover the law of S_T at time T from call prices $C(K)$.... Skorokhod Embedding Problem... David's talks...

Variance Options

- We may typically suppose a model for (discounted) asset prices of the form:

$$\frac{dS_t}{S_t} = \sigma_t dW_t,$$

where W_t a Brownian motion.
- the volatility, σ_t, is a locally bounded, progressively measurable process
- Want to consider options on variance.
- For example, a variance swap pays:

$$\int_0^T \left(\sigma_t^2 - \bar{\sigma}^2\right) dt$$

where $\bar{\sigma}$ is the 'strike'. Dupire (1993) and Neuberger (1994) gave a simple replication strategy for such an option. (More recently, Davis-Obłój-Ramal, 2013).

Hedge of Variance Swap

- Dupire's hedge: Itô implies

$$d\,(\ln S_t) = \sigma_t\, dW_t - \frac{1}{2}\sigma_t^2\, dt$$

- Hold portfolio short 2 contracts paying $\ln(S_T)$, long $2/S_t$ units of asset
- At time T, portfolio will be worth (up to constant) $\int_0^T \sigma_t^2\, dt$
- Note that the only modelling assumption here is that the volatility process exists!
- Note also that $\langle \ln S \rangle_T = \int_0^T \sigma_t^2\, dt$, where $\langle \cdot \rangle_t$ is quadratic variation.

Variance Options

- A variance call is an option paying:

$$\left(\langle \ln S\rangle_T - K\right)_+$$

- More general options of the form: $F(\langle \ln S\rangle_T)$.
- E.g.: volatility swap, payoff:

$$\sqrt{\langle \ln S\rangle_T} - K.$$

- More generally, can consider payoffs dependent on weighted realised variance:

$$RV_T^\lambda = \int_0^T \lambda(S_t)\, d\langle \ln S\rangle_t = \int_0^T \lambda(S_t)\sigma_t^2 dt.$$

- E.g.: options on corridor variance or a gamma swap:

$$\int_0^T \mathbf{1}_{\{S_t\in[a,b]\}}\, d\langle \ln S\rangle_t\,, \quad \int_0^T S_t\, d\langle \ln S\rangle_t\,.$$

Options on (weighted) realised variance

- Let $\lambda(x)$ be a strictly positive, continuous function, $\tau_t := RV^{\lambda}_T = \int_0^t \lambda(S_s)\sigma_s^2\, ds$ and A_t such that $\tau_{A_t} = t$.
- Then $\widetilde{W}_t = \int_0^{A_t} \sigma_s \lambda(S_s)^{1/2}\, dW_s$ is a BM w.r.t. $\widetilde{\mathcal{F}}_t = \mathcal{F}_{A_t}$, and if we set $\widetilde{X}_t = S_{A_t}$, we have:

$$d\widetilde{X}_t = \widetilde{X}_t \lambda(\widetilde{X}_t)^{-1/2}\, d\widetilde{W}_t.$$

- $\widetilde{X}_t$ is now a diffusion on natural scale
- $(\widetilde{X}_{\tau_T}, \tau_T) = (S_T, RV^{\lambda}_T)$
- Knowledge of $\mathcal{L}(S_T) \implies \mathcal{L}(\widetilde{X}_{\tau_T})$.

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Variance Call

- This suggests finding lower/upper bound on price of variance call (say) with given call prices is equivalent to:

  minimise/maximise: $\mathbb{E}(\tau - K)_+$ subject to: $\mathcal{L}(\widetilde{X}_\tau) = \mu$

  where μ is a given law.
- Are there Skorokhod Embeddings which do this?

Root's Construction

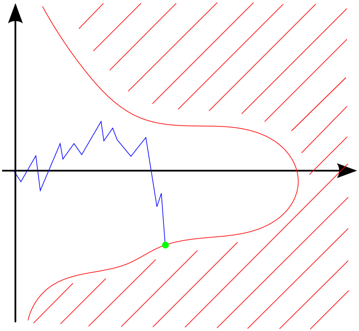
- $B \subseteq \mathbb{R} \times \mathbb{R}_+$ a barrier if:

  $$(x, t) \in B \implies (x, s) \in B$$

  for all $s \geq t$
- Given μ, and $D = B^C$, exists stopping time

  $$\tau_D = \inf\{t \geq 0 : (\widetilde{X}_t, t) \notin D\}$$

  which is an embedding.
- Minimises $\mathbb{E}(\tau - K)_+$ over all (UI) embeddings
- Root (1969)
- Rost (1976)



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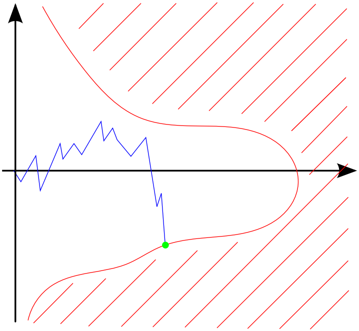

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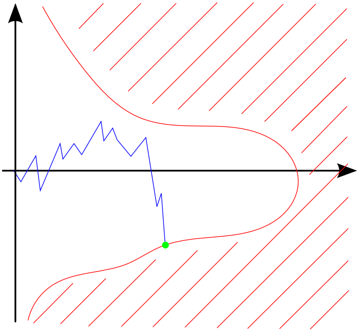

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Rost's Construction

- $B \subseteq \mathbb{R} \times \mathbb{R}_+$ a reversed barrier if:

$$(x, t) \in B \implies (x, s) \in B$$

for all $s \leq t$

- Given μ, and $D = B^C$, exists a stopping time

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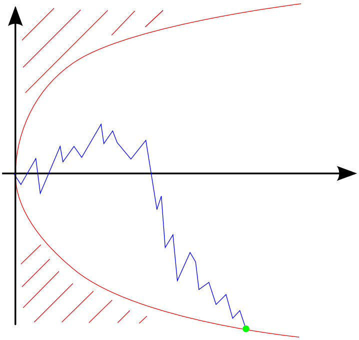

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- Chacon (1985)
- McConnell (1991)
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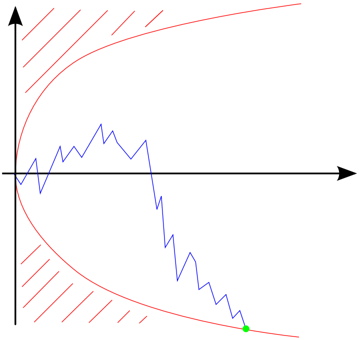

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  where μ is a given law.
- These are (essentially) the problems solved by Root's and Rost's Barriers!
- Rost (1971) proved the existence of a filling scheme stopping time for a general class of processes. Chacon (1985) showed that the filling scheme was indeed a reversed barrier under some assumptions on the process, and proved optimality.
- The connection to Variance options has been observed by a number of authors: Dupire ('05), Carr & Lee ('09), Hobson ('09).

Questions

Question

This known connection leads to two important questions:

1. *How do we find the Root/Rost stopping times?*
2. *Is there a corresponding hedging strategy?*

- Dupire gave a connected free boundary problem for Root
- In C. & Wang, gave a variational characterisation of Root's barrier & construction of optimal strategy; Oberhauser & dos Reis gave characterisation as viscosity solution. Key step in construction for Root: classical results on existence of solution.

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Rost's Solution: Class of possible processes

Suppose

$$d\widetilde{X}_t = \widetilde{X}_t \lambda(\widetilde{X}_t)^{-1/2}\, d\widetilde{W}_t,$$

where λ is in the set $\mathcal{D} \subseteq C(I;\mathbb{R})$ such that

- $\lambda(x)$ is strictly positive,
- $\widetilde{X}$ is a regular diffusion on I,
- with transition density $p(t,x,y)$ with respect to Lebesgue
- such that, for any $x_0 \in I$, $c > 0$, open set A containing x_0 and $\varepsilon > 0$, there exists $\delta > 0$ such that

  $$|\,(p(t,x,x_0) - p(s,x,x_0))\, x_0^2 \lambda(x_0)^{-1}| < \varepsilon$$

  whenever $|s - t| < \delta$ and either $x_0 \notin A$ or $t > c$.
  (Chacon's Equicontinuity condition)

Rost's Solution

Theorem (Rost & Chacon)

Suppose μ and ν are probability measures on I with $\nu \leq_{cx} \mu$, and $\lambda \in \mathcal{D}$ with $\widetilde{X}_0 \sim \nu$:

1. *if μ and ν have no mass in common, then there exists a reversed barrier D such that $\widetilde{X}_{\tau_D} \sim \mu$;*
2. *if μ and ν have mass in common, then (on a possibly enlarged probability space) there exists a random variable $S \in \{0, \infty\}$, and reversed barrier D, such that $\widetilde{X}_{\tau_D \wedge S} \sim \mu$.*

Moreover, in both cases, the resulting embedding maximises $\mathbb{E}F(\sigma)$ over all stopping times σ with $\widetilde{X}_\sigma \sim \mu$ and $\mathbb{E}\sigma = \mathbb{E}\tau_D \wedge S < \infty$, for any convex function F on $[0, \infty)$.

See also Beiglböck & Huesmann (2013) for a promising alternative approach to existence!

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Characterising the Barrier

Suppose $I = (0, \infty)$, $\lambda \in C^1(I) \cap \mathcal{D}$ and, $|\lambda(x)^{-1}|$ and $|\lambda'(x)\lambda(x)^{-2}x|$ are bounded on $(0, \infty)$.

Theorem

Suppose D is Rost's reversed barrier. Then $u(x,t) = \mathrm{U}_\mu(x) + \mathbb{E}^\nu \left| x - \widetilde{X}_{t \wedge \tau_D \wedge S} \right|$ *is the unique bounded viscosity solution to:*

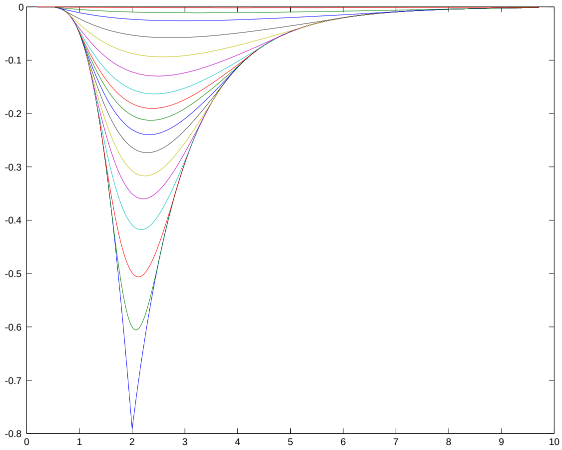
$$\frac{\partial u}{\partial t}(x,t) = \left(\frac{\sigma(x)^2}{2} \frac{\partial^2 u}{\partial x^2}(x,t) \right)_+$$

$$u(0,x) = \mathrm{U}_\mu(x) - \mathrm{U}_\nu(x).$$

Moreover, given a solution u, a reversed barrier D which solves the SEP can be recovered by $D = \{(x,t) : u(x,t) > u(0,t)\}$.

See also Oberhauser & dos Reis (2013).

Computing the Barrier



- Optimal stopping interpretation:
  $u(x,t) = \sup_{\tau \le t} \mathbb{E}^x \left[U_\mu(\widetilde{X}_\tau) - U_\nu(\widetilde{X}_\tau) \right]$

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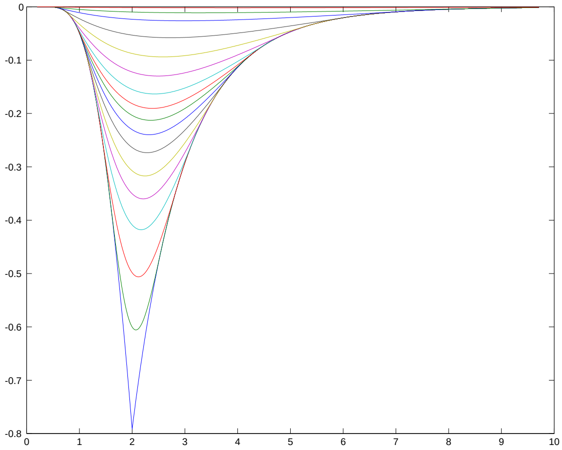

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Optimality of Rost's Barrier

Chacon's Result

Given a function F which is convex, increasing, Rost's (reversed) barrier solves:

$$\begin{array}{ll} \text{maximise} & \mathbb{E}F(\widetilde{X}_\tau) \\ \text{subject to:} & \widetilde{X}_\tau \sim \mu \\ & \tau \text{ a stopping time} \end{array}$$

Want:

- A simple proof of this…
- …that identifies a 'financially meaningful' hedging strategy.

For simplicity, consider the case where S_0 is non-random.

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Optimality

Write $f(t) = F'(t)$, define

$$M(x,t) = \mathbb{E}^{(x,t)} f(\tau_D),$$

and fix $T > 0$. Write $\sigma(x) = x\lambda(x)^{-1/2}$. Then we set

$$Z_T(x) = 2\int_{S_0}^{x}\int_{S_0}^{y} \frac{M(z,T)}{\sigma^2(z)}\,dz\,dy,$$

so that in particular, $Z_T''(x) = 2\sigma^2(x)M(x,T)$. And finally, let:

$$G_T(x,t) = F(T) - \int_t^T M(x,s)\,ds - Z_T(x)$$

$$H_T(x) = \int_{R(x)\wedge T}^{T} [M(x,s) - f(s)]\,ds + Z_T(x)$$

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Optimality

Then there are two key results:

Proposition

For all $(x, t, T) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$*:*

$$G_T(x,t) + H_T(x) \geq F(t) \textit{ in } D,$$
$$G_T(x,t) + H_T(x) = F(t) \textit{ in } D^C.$$

Note that it follows that the reversed barrier stopping time attains equality.

Optimality

Define $Q(x) = \int_{S_0}^{x} \int_{S_0}^{y} 2\sigma(z)^{-2}\, dz\, dy$

Lemma

Suppose f is bounded and for any $T > 0$, $\left(Q(\widetilde{X}_t);\, 0 \le t \le T\right)$ *is UI. Then for any* $T > 0$, *the process*

$$\left(G_T(\widetilde{X}_{t\wedge\tau_D}, t \wedge \tau_D);\, 0 \le t \le T\right) \quad \text{is a martingale,}$$

and

$$\left(G_T(\widetilde{X}_t, t);\, 0 \le t \le T\right) \quad \text{is a supermartingale.}$$

Note that we only have martingale properties up to T!

Optimality

Theorem

Suppose τ_D is Rost's solution to the SEP, and for all $T > 0$, $\left\{ Q(\widetilde{X}_t); 0 \leq t \leq T \right\}$ is a uniformly integrable family. Then τ_D maximises $\mathbb{E}F(\tau)$ over $\tau : \widetilde{X}_\tau \sim \mu$.

- This is just Chacon's optimality result.
- Note that the UI condition is easily checked when $\sigma(x) = x$.

Optimality: sketch of proof

So for any solution τ to the Skorokhod embedding problem, (if we assume also $\tau, \tau_D \leq T$!)

$$\mathbb{E}G_T(\widetilde{X}_\tau, \tau) + \mathbb{E}H_T(\widetilde{X}_\tau) \geq \mathbb{E}F(\tau).$$

But $\mathbb{E}H_T(\widetilde{X}_\tau)$ depends only on the law of $\widetilde{X}_\tau$, and

$$\mathbb{E}G_T(\widetilde{X}_\tau, \tau) \leq \mathbb{E}G_T(\widetilde{X}_{\tau_D}, \tau_D) = G_T(\widetilde{X}_0, 0).$$

In addition, we get equality, $G_T(\widetilde{X}_{\tau_D}, \tau_D) + H_T(\widetilde{X}_{\tau_D}) = F(\tau_D)$, for the Rost stopping time, so $\mathbb{E}F(\tau) \leq \mathbb{E}F(\tau_D)$.

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Optimality

The additional T in the construction means that we cannot construct a pathwise inequality for all cases, even though we can prove optimality in general by a limiting argument.

However the functions G_T, H_T, Z_T can be interpreted in the limit provided we can find $\alpha > 1$ such that for t large:

$$C \geq F'(t) \geq C - O(t^{-\alpha}).$$

In this case, we do indeed have a pathwise inequality, and can derive a pathwise inequality

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Hedging Strategy

Since $G_T(\widetilde{X}_t, t)$ is a supermartingale, (and if things are well-behaved there is a trading strategy which super-replicates $G_T(\widetilde{X}_t, t)$:

$$G_T(S_t, \langle \ln S \rangle_t) \leq \int_0^t \frac{\frac{\partial G_T}{\partial x}(S_r, \langle \ln S \rangle_r)}{\sigma_r^2} \, dS_r$$

and $H_T(X_t)$ can be replicated by holding a suitable portfolio of the traded calls.

Moreover, in the case where $\tau = \tau_D$, we get equality.

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Forward Starting options

More generally, we can consider forward starting options, whose payoff is $F(\langle \ln S\rangle_{t_2} - \langle \ln S\rangle_{t_1})$ if we know the call prices at times t_1, t_2.

Construct the barrier where the initial law is now non-trivial: use the law at time t_1 instead.

As well as the portfolio of calls at time t_2, and the dynamic trading strategy as above, we must also be able to replicate $G(\widetilde{X}_{t_1}, 0)$. However, this can be done using the calls with maturity t_1.

Practical implementation

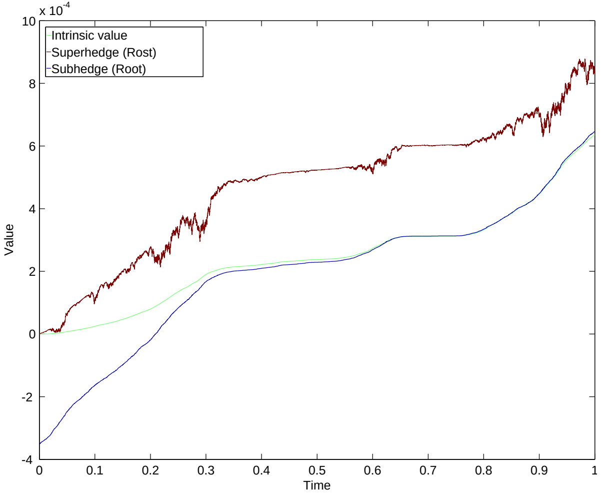
Well-known results on viscosity solutions mean that we can use standard discretisation methods (Barles-Souganidis, c.f. Oberhauser & dos Reis) for PDEs to find u, and thus the reversed barrier.

In fact, with a little extra work, we can even use implicit methods — for Rost, this seems necessary (lots of detail at the start!)

Can then compute the hedging strategies, and the upper and lower price bounds.

Numerical Implementation

- Payoff: $F'(t) = 2(t \wedge K)$, $K \approx 0.2$. Under the true model.



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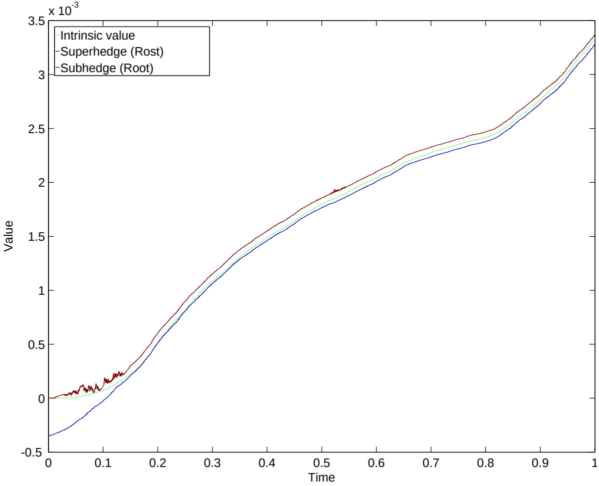

Heston-Nandi model

The Heston model is given (under the risk-neutral measure) by:

$$dS_t = rS_t\,dt + \sqrt{v_t}S_t\,dB_t,$$
$$dv_t = \kappa(\theta - v_t)\,dt + \xi\sqrt{v_t}\,d\widetilde{B}_t,$$

where B_t and $\widetilde{B}_t$ are Brownian motions with correlation ρ. The Heston-Nandi model is the restricted case where $\rho = -1$, and so $\widetilde{B}_t = -B_t$. Note that $v_t = \sigma_t^2$ in our previous notation, so we are interested in options on $\int_0^T v_t\,dt$

Heston-Nandi and Barrier stopping times

Using Itô's Lemma, we know

$$d(\log(\mathrm{e}^{-rt}S_t)) = -\frac{1}{2}v_t\,dt + \sqrt{v_t}\,dB_t$$
$$= \left(\frac{\kappa\theta}{\xi} - \left(\frac{\kappa}{\xi} + \frac{1}{2}\right)v_t\right)\,dt - \frac{1}{\xi}dv_t.$$

Solving, we see that

$$\log\left(\frac{\mathrm{e}^{-rT}S_T}{S_0}\right) = \frac{1}{\xi}(v_0 - v_T) + \frac{\kappa\theta}{\xi}T - \left(\frac{\kappa}{\xi} + \frac{1}{2}\right)\int_0^T v_t\,dt.$$

Since v_T is mean reverting, $(v_T - v_0) \approx (\theta - v_0)$ will be comparatively small for large T. In this case, we can write:

$$\int_0^T v_t\,dt \approx R_T(\mathrm{e}^{-rT}S_T) = \left(\frac{\kappa}{\xi} + \frac{1}{2}\right)^{-1}\left[\frac{\kappa\theta}{\xi}T + \log\left(\frac{S_0}{\mathrm{e}^{-rT}S_T}\right)\right]$$

Heston-Nandi and Barrier stopping times

Using Itô's Lemma, we know

$$\begin{aligned} d(\log(\mathrm{e}^{-rt}S_t)) &= -\frac{1}{2}v_t\,dt + \sqrt{v_t}\,dB_t \\ &= \left(\frac{\kappa\theta}{\xi} - \left(\frac{\kappa}{\xi} + \frac{1}{2}\right)v_t\right)\,dt - \frac{1}{\xi}dv_t. \end{aligned}$$

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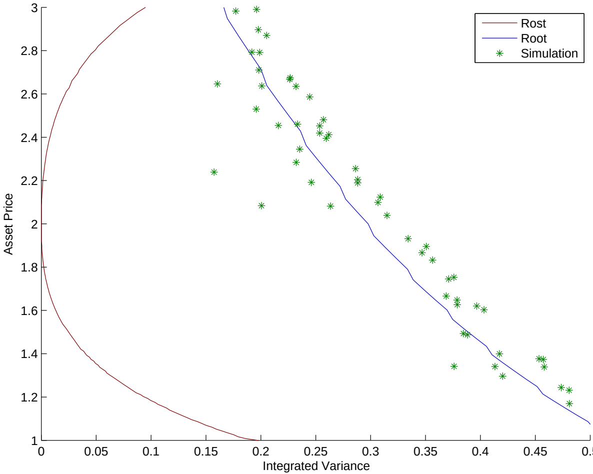
Heston-Nandi and barriers

- Samples from the Heston-Nandi model, and the corresponding barrier function. And an uncorrelated Heston model



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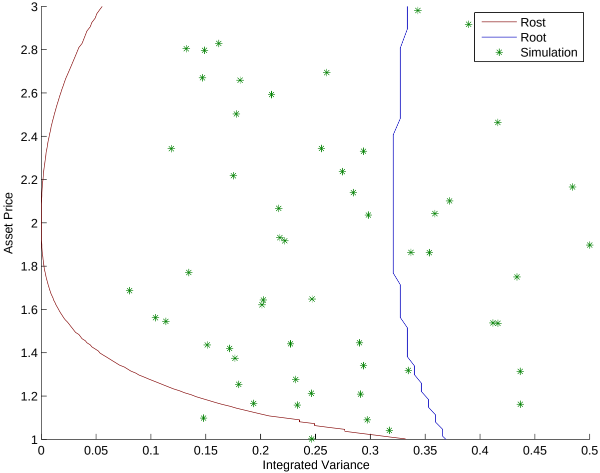

Large T asymptotics

Theorem

Let $M > 0$ and suppose $\xi, \theta, \kappa, r > 0, \xi \neq 2\kappa$ are given parameters of a Heston-Nandi model, $\mathbb{Q}^{HN}$. Suppose $\mathcal{Q}_T$ is the class of models $\mathbb{Q}$ satisfying $\mathbb{E}^{\mathbb{Q}^{HN}}(S_T - K)_+ = \mathbb{E}^{\mathbb{Q}}(S_T - K)_+$ for all $K \geq 0$.

Then there exists a constant κ, depending only on M and the parameters of the Heston-Nandi model, such that for all convex, increasing functions $F(t)$ with suitably smooth derivative $f(t) = F'(t)$ such that $f(t), f'(t) \leq M^*$, and for all $T \geq 0$

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Heston-Nandi is asymptotically optimal.

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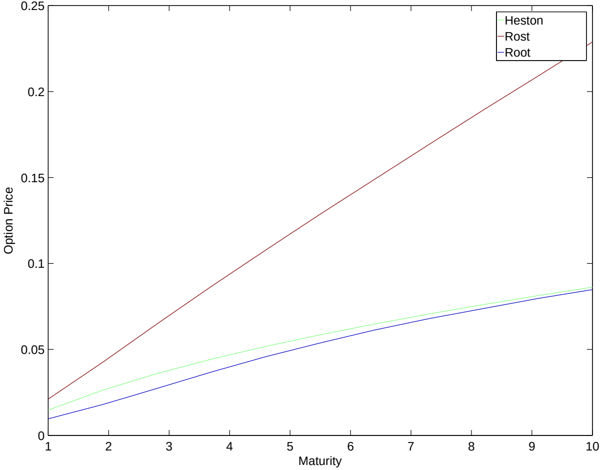
Numerical Evidence

- The theorem is rather weak — numerical evidence suggests there is more to be said:



Conclusion

- Model-free lower & upper bounds on variance options $\sim$ finding Root & Rost's barriers
- Can characterise (and compute) the barriers
- Explicit construction of robust super/sub-hedging strategies
- Heston-Nandi model is 'asymptotically extreme' for variance options